Proof Of Maxwell's Equations

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ABSTRACT

Aims. Our goal here is to rigorously prove the equations of James Clerk Maxwell using line integrals and vector field rules.Maxwell's equations allow us to reconstruct the electric and magnetic fields from the charge and current densities. They are fundamental to the subject of electricity and magnetism and provide a fitting tribute to the power of the theorems of Stokes and Gauss.
Methods. Our proof will utilise vector analysis, Green's Theorem, and Gauss's theorem to prove the equations. We shall using established laws of physics from this proof and assume that the reader has an understanding of multi-variable calculus
Results. The First Equation was found to be the differential form of Gauss's law is equivalent to Coulomb's law, the second equation states that there are no sources of magnetic field except currents; that is, there are no magnetic monopoles. The third equation expresses the dependence of the magnetic field on the displacement current density, or rate of change of electric field, and on the conduction current density, or rate of motion of charge. The final equation is is Faraday's law of induction
Conclusions. The equations were proven by using the rules of line integrals, vector fields, and Gauss's theorem.

1. Introduction

Maxwell's Equations are defined as the following.

Gauss's Law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

This is the differential form of Gauss's Law
No Magnetic Monopoles

$$\nabla \cdot \mathbf{B} = 0$$

The intuitive content of this equation is often expressed by saying that "magnetic monopoles" do not exist.
Faraday's Law
Michael Faraday observed empirically that the change in magnetic flux across a surface S equals the electromotive force around the boundary C of the surface. This relation can be written as

$$\nabla \cdot \mathbf{E} = -\frac{\delta \mathbf{B}}{\delta t}$$

Ampére's law generalised

$$\nabla \cdot \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\delta \mathbf{E}}{\delta t}$$

This is known as the displacement current density, was first postulated by James Clerk Maxwell in order to generalised Ampére's law to the non-static case.

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2. Proof Of Gauss's Law

Assume that There Exists and electrical field designated by **E**, the flux of the electrical field across a closed surface S shall be given by the below.

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

We then apply Gauss's Theorem

$$\iiint_D \nabla \cdot \mathbf{E} dV$$

D being the region enclosed by S

If the electric field **E** is determined by a single point charge of q coulombs (coulombs shall henceforth be designated as C) located at the origin, then **E** is given by the below.

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{x}}{||\mathbf{x}||^{3\prime}}$$

We can let **x** be represented by the vector $\begin{bmatrix} x & y & z \end{bmatrix}$ in mks units, **E** is measured in volts/meter the constant ϵ_0 is known as the **permittivity of free space**; its value (in mks units) is $8.854 \times 10^{-12} \frac{C^2}{N \cdot m^2}$

For the electric field equation we can readily verify that $\nabla \cdot \mathbf{E} = 0$ wherever **E** is defined. From the formulae given to us by Gauss's theorem if S is any surface that does *not* enclose the origin, then the flux of **E** across S is zero.

But now a question arises: How do we calculate the flux of the electric field across surfaces that *do* enclose the origin? The trick is to find an appropriate way to exclude the origin from consideration. To that end, first suppose that S_b denotes a sphere of radius b entered at the origin. ($S_b = x^2 + y^2 + z^2 = b^2$) Then the outward unit normal to S_b is given by

$$\mathbf{n} = \frac{\begin{bmatrix} x & y & z \end{bmatrix}}{b} = \frac{\mathbf{x}}{b}$$

we then apply the first equation.

$$\iint_{S_b} \mathbf{E} \cdot dS = \frac{4}{4\pi\epsilon_0} \iint_{S_b} \frac{\mathbf{x}}{||\mathbf{x}||^3} \cdot \frac{\mathbf{x}}{b} d\mathrm{S}$$

$$||\mathbf{x}|| = b| \iint_{S_b} \mathbf{E} \cdot dS = \frac{q}{4\pi\epsilon_0} \iint_{S_b} \frac{\mathbf{x}}{b^3} \cdot \frac{\mathbf{x}}{b} dS = \iint_{S_b} \frac{||\mathbf{x}||^2}{b^4} dS$$

$$= \frac{q}{4\pi\epsilon_0} \iint_{S_b} \frac{b^2}{b^4} dS = \frac{q}{4\pi\epsilon_0 b^2} \iint_{S_b} dS$$

$$\frac{q}{4\pi\epsilon_0 b^2}(4\pi b^2) = \frac{q}{\epsilon_0}$$

Now, suppose S is any surface enclosing the origin. Let S_b be a small sphere centred at the origin and contained inside S. Let D be the solid region $\in \mathbb{R}^3$ between S_b & S. Note that $\nabla \cdot \mathbf{E} = 0$ throughout D, since D doesn't contain the origin.

If we orient the equation $\delta D = S \cup S_b$ with normal vectors that point away from D we obtain the below

$$\iint_S \mathbf{E} \cdot dS - \iint_{S_b} \mathbf{E} \cdot dS = \iiint_D \nabla \cdot \mathbf{E} dV = 0$$

$$\therefore \iint_S \mathbf{E} \cdot dS = \frac{q}{\epsilon_0} \textit{ for any surface that encloses the origin.}$$

We can then modify the equation for $\mathbf{E}$ and we can show that the above holds true for any closed surface containing a single point charge of q coulombs located anywhere. We can adapt the arguments just given to accommodate the case of n discrete point charges. For $i = 1, \ldots, n$, suppose a point charge of q_i coulombs is located at position $\mathbf{r}_i$. The electric field $\mathbf{E}$ for this configuration is below.

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} qi \frac{\mathbf{x} - \mathbf{r}}{||\mathbf{x} - \mathbf{r}_i||^3}$$

For $\mathbf{E}$ from the given equation, we can calculate that $\nabla \cdot \mathbf{E} = 0$ with the exception of $\mathbf{x} = \mathbf{r}_i$ if S is any closed, piecewise smooth, outwardly-oriented surface containing the charges, then we may use Gauss's theorem to find the flux of $\mathbf{E}$ across S by taking n small spheres $S_1, S_2, \ldots, S_n$ each enclosing a single point charge. If D is the region inside S but outside all the spheres, we have, by choosing appropriate orientations and using Gauss's theorem, the below.

$$\iint_S \mathbf{E} \cdot dS - \sum_{i=1}^{n} \iint_{S_i} \mathbf{E} \cdot dS = \iint_{\delta D} \mathbf{E} \cdot dS = \iiint_D \nabla \cdot \mathbf{E} dV = 0$$

Given that $\nabla \cdot \mathbf{E} = 0$ w.r.t. D

$$\iint_S \mathbf{E} \cdot dS = \sum_{i=1}^{n} \iint_{S_i} \mathbf{E} \cdot dS = \frac{1}{\epsilon_0} \sum_{i=1}^{n} qi = \frac{1}{\epsilon_0}$$

To establish Gauss's law, consider the case not of an electric field determined by discrete point charges, but rather of one determined by a continuous charge distribution given by a charge density $\rho(\mathbf{x})$ The total change over a region D in space is given by

$$\iiint_D \rho(\mathbf{x})\, dV$$

And if we apply this to our electric field formula we get the below.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint_D \rho(\mathbf{x}) \frac{\mathbf{r} - \mathbf{x}}{||\mathbf{r} - \mathbf{x}||^3} dV$$

This the integration occurs with respect to the variables in $\mathbf{x}$. This intergral used to define $\mathbf{E}$(r) converges at points $\mathbf{r} \in D$ where $\rho(\mathbf{r}) \neq 0$, because because at such points the triple integral is improper.

Thus the integral form of Gauss's Law is

$$\iint_S \mathbf{E} \cdot dS = \frac{1}{\epsilon_0} \iiint_D \rho\, dV$$

Given that $S = \delta D$ If we apply Gauss's theorem to the left side of the formula, we find that.

$$\iiint_D \nabla \cdot \mathbf{E}\, dV = \frac{1}{\epsilon_0} \iiint_D \rho\, dV$$

Given that D is an apbitrary region it may be *shrunk* to a point, Thus we can conclude that.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

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3. Proof that Magnetic Monopoles Do not exist

A moving charged particle generates a magnetic field. To be specific, if a point charge of q coulombs is at position $\mathbf{r}_0$ and is moving with velocity $\mathbf{v}$, then the magnetic field it induces is given below.

$$\mathbf{B}(\mathbf{r}) = \left(\frac{\mu_0 q}{4\pi}\right)\left(\frac{\mathbf{v} \times (\mathbf{r}_{-}\mathbf{r}_0)}{||(\mathbf{r} - \mathbf{r}_0)||^3}\right)$$

$\mathbf{B}$ is measured in teslas. The constant μ_0 is known as the **permeability of free space**

$$\mu_0 = 4\pi \times 10^{-7} \frac{N}{amp^2}$$

In the case of a magnetic field that arises from a continuous, charged medium wire), rather than from a single moving charge, we replace q by a suitable charge density function ρ and the velocity of a single particle by the velocity vector field **v** of the charges. Then we define the current density field **J** by the below.

$$\mathbf{J}(\mathbf{x}) = \rho(\mathbf{x})\mathbf{v}(\mathbf{x})$$

In place of our initial definition of $\mathbf{B}(\mathbf{r})$ we use the following definition for the magnetic field resulting from moving charges in a region D in space:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\iiint_D \rho(\mathbf{x})\mathbf{v}(\mathbf{x}) \times \frac{\mathbf{r}-\mathbf{x}}{||\mathbf{r}-\mathbf{x}||^3}\, dV$$

$$= \frac{\mu_0}{4\pi}\iiint_D \mathbf{J}(\mathbf{x}) \times \frac{\mathbf{r}-\mathbf{x}}{||\mathbf{r}-\mathbf{x}||^3}\, dV$$

The variables constituting **x** it is not obvious that the integrals are convergent if $\mathbf{r} \in D$

Before continuing our calculations,we comment further regarding the current density field **J**. The vector field **J** at a point is such that its magnitude is the current per unit area at that point, and its direction is that of the current flow. It is not hard to see then that the total current I across an oriented surface S is given by the flux of **J**; that is.

$$I = \iint_S \mathbf{J} \cdot dS$$

Revisiting **B**, we show that it can be identified as a curl of vector field **A**, firstly we shall determine this by direct calculation.

$$\nabla_r\left(\frac{1}{||\mathbf{r}-\mathbf{x}||}\right) = -\frac{\mathbf{r}-\mathbf{x}}{||\mathbf{r}-\mathbf{x}||^3}$$

$$\implies \mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\iiint_D \mathbf{J}(\mathbf{x}) \times \nabla_r\left(\frac{1}{||\mathbf{r}-\mathbf{x}||}\right) dV$$

Using the identity where f is a scalar field and **F** a vector field

$$\nabla \times (f\mathbf{F}) = (\nabla \times \mathbf{F})f - \mathbf{F} \times \nabla f$$

$$\implies \mathbf{F} \times \nabla f = (\nabla \times \mathbf{F})f - \nabla \times (f\mathbf{F})$$

$$\therefore \mathbf{J}(\mathbf{x}) \times \nabla_r\left(\frac{1}{||\mathbf{r}-\mathbf{x}||}\right) = \frac{(\nabla_r \times \mathbf{J}(\mathbf{x}))}{||\mathbf{r}-\mathbf{x}||} - \nabla_r \times \frac{\mathbf{J}(\mathbf{x})}{||\mathbf{r}-\mathbf{x}||} = -\nabla_r \times \frac{\mathbf{J}(\mathbf{x})}{||\mathbf{r}-\mathbf{x}||}$$

Given that $\mathbf{J}(\mathbf{x})$ is independent of **r**

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\iiint_D \nabla \times \frac{\mathbf{J}(\mathbf{x})}{||\mathbf{r}-\mathbf{x}||}\, dV = \frac{\mu_0}{4\pi}\nabla_r \times \iiint_D \frac{\mathbf{J}(\mathbf{x})}{||\mathbf{r}-\mathbf{x}||}\, dV$$

None of the variables of integration are contained in **r**

$$\therefore \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}), \iff \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\iiint_D \frac{\mathbf{J}(\mathbf{x})}{||\mathbf{r}-\mathbf{x}||}\, dV$$

$$\therefore \nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A})$$

From theorem 4.4 from Chapter 3 we know this.

$$\therefore \nabla \cdot \mathbf{B} = 0$$

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Therefore we have proven that magnetic monopoles are impossible.

4. Ampére's law

4.1. Differential Form of Ampére's law in the static case.

If C is a closed loop enclosing a current I, then Ampére's law says that, up to a constant, the current through the loop is equal to the circulation of the magnetic field around C.

$$\oint_C \mathbf{B} \cdot ds = \mu_0 I$$

We can assume that C is oriented so that C and I are related by a right-hand rule, that is, that they are related in the same way that the orientation of C and the normal to any surface S that C bounds are related in Stokes's theorem.

As we proved in the prior section the total current can be rewritten as below. Assuming that S is a surface bounded by C

$$\oint_C \mathbf{B} \cdot ds = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{S}$$

Then we apply Stokes's theorem.

$$\iint_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{S}$$

$$\implies \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

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4.2. Displacement current density, Or Generalisation of Ampére's law

In the event that the magnetic and electric fields are time varying, we need to make some modifications. From the Equation of continuity we know that.

$$\nabla \cdot \mathbf{J} = -\frac{\delta\rho}{\delta t}$$

$$\implies \nabla(\nabla \times \mathbf{B}) = \nabla \cdot (\mu_0 \mathbf{J}) = -\mu_0 \frac{\delta\rho}{\delta t}$$

If we let **B** be of class C^2 we must then have $\nabla \cdot (\nabla \times \mathbf{B}) = 0$ for all caeses where ρ is not constant with respect to time. We can then modify the differential form of Ampére's law by adding the extra term.
We know from Gauss's Law that $\frac{\delta\rho}{\delta t} = \epsilon_0 \nabla \cdot \frac{\delta \mathbf{E}}{\delta t}$ Then we can substitute **J** with $\mathbf{J} + \epsilon_0 \frac{\delta \mathbf{E}}{\delta t} | \nabla \cdot (\nabla \times \mathbf{B}) = 0$

$$\therefore \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\delta \mathbf{E}}{\delta t}$$

■ Thus we have proven the general form of Ampére's law.

5. Faraday's law

Michael Faraday observed empirically that the change in magnetic flux across a surface S equals the electromotive force around the boundary C of the surface. This relation can be written as.
We can let there exist $\Phi | \Phi(t) = \iint_S \mathbf{B} \cdot d\mathbf{S}$

$$\frac{d\Phi}{dt} = \oint_C \mathbf{E} \cdot d\mathbf{s}$$

We can then apply our old friend Stokes's Theorem to the line integral and find the below.

$$\oint_c \mathbf{E} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{E} \cdot d\mathbf{s}$$

And we know that.

$$\frac{d\Phi}{dt} = \frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S} = \iint_S \frac{\delta \mathbf{B}}{\delta t} \cdot d\mathbf{S}$$
$$\implies -\iint_S \frac{\delta \mathbf{B}}{\delta t} \cdot d\mathbf{S} = \iint_S \nabla \times \mathbf{E} \cdot d\mathbf{S}$$

Given that this is an arbitrary surface we can conclude the below.

$$\therefore \nabla \times \mathbf{E} = -\frac{\delta \mathbf{B}}{\delta t}$$

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References

[1] Colley, Susan Jane. *Vector Calculus*(4th ed, P.512-518) Boston, Massachusetts, Pearson, 2012
[2] Purcell, Edward Millis, and David Morin. *Electricity and Magnetism* (3rd ed, P.436-438) Cambridge, Cambridge University Press, 2018.