A16729W1

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B5. GENERAL RELATIVITY

TRINITY TERM 2022

Thursday, 16 June, 2.30 pm – 4.30 pm

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

The use of approved calculators is permitted.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Page 2 contains physics formulae and data for this paper.
The questions start on page 3.

Expressed in terms of the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$, the Riemann tensor has components

$$R^{\gamma}{}_{\alpha\beta\delta} = \frac{\partial}{\partial x^{\delta}}\Gamma^{\gamma}_{\alpha\beta} - \frac{\partial}{\partial x^{\beta}}\Gamma^{\gamma}_{\alpha\delta} + \Gamma^{\epsilon}_{\alpha\beta}\Gamma^{\gamma}_{\delta\epsilon} - \Gamma^{\epsilon}_{\alpha\delta}\Gamma^{\gamma}_{\beta\epsilon}.$$

The Ricci tensor is defined as $R_{\alpha\beta} \equiv -R^{\gamma}{}_{\alpha\beta\gamma}$. When expressed in a basis in which the metric tensor $g_{\alpha\beta}$ is diagonal, it can be written as

$$R_{\alpha\beta} = \frac{1}{2}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}}\ln|g| - \frac{\partial\Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} + \Gamma^{\delta}_{\alpha\gamma}\Gamma^{\gamma}_{\beta\delta} - \frac{1}{2}\Gamma^{\gamma}_{\alpha\beta}\frac{\partial}{\partial x^{\gamma}}\ln|g|,$$

where $|g|$ is the modulus of the determinant of $g_{\alpha\beta}$.

The Einstein equation is

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = -\frac{8\pi G}{c^4}T_{\alpha\beta},$$

where $R \equiv g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci scalar, Λ is the cosmological constant and $T_{\alpha\beta}$ is the energy–momentum tensor.

For the line element

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right]$$

the Einstein equation can be written as the pair of equations

$$\begin{aligned}\ddot{R} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{1}{3}\Lambda c^2 R,\\ \dot{R}^2 &= \frac{8\pi G}{3}\rho R^2 + \frac{1}{3}\Lambda c^2 R^2 - c^2 k,\end{aligned}$$

where ρ and p are the rest-frame density and pressure, respectively.

1. The Riemann curvature tensor can be written as:

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right) + g_{\eta\sigma}\left(\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right).$$

(a) State its symmetries without proving them. [4]

(b) It obeys the Bianchi identities that are central to General Relativity, and can be written in this form:

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0.$$

What is meant by the ";" symbol? From the Bianchi identities, derive:

$$\left(R^{\mu\nu} - g^{\mu\nu}\frac{R}{2}\right)_{;\mu} = 0.$$

Clearly define all the symbols used and explain why it is useful to construct such a tensor with vanishing divergence. [7]

(c) Consider the following form of the field equations:

$$R_{\mu\nu} - g_{\mu\nu}\frac{R}{2} = CT_{\mu\nu}.$$

Note the lowered indices. Show that this can be written as:

$$R_{\mu\nu} = CS_{\mu\nu}$$

where $S_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu}T/2$, and C is a constant. [3]

(d) The Newtonian limit implies a non-relativistic source particle producing a weak and static gravitational field. Write down the definition of the Ricci tensor $R_{\mu\nu}$ and specifically the R_{00} term. At the Newtonian limit, show that $R_{00} = \frac{1}{2}C\rho c^2$, where ρ is the Newtonian mass density and c the speed of light. Write down the Riemann tensor in the weak field limit, and then R_{00} specifically. Simplify R_{00} by assuming the static limit, to show:

$$R_{00} = \frac{1}{2}\nabla^2 g_{00}.$$

[7]

(e) Finally, define C such that Poisson's equation for gravity is satisfied in the Newtonian limit, taking $g_{00} \simeq -(1 + 2\Phi/c^2)$. [4]

2. (a) For a curved spacetime, show how the metric tensor $g_{\mu\nu}$ can be expressed in terms of a suitable constant tensor and relations involving a set of locally inertial coordinates ξ^α. [2]

(b) Write down an equation defining the covariant derivative of a vector A^λ, and a clear explanation of its usefulness in a curved spacetime. In your definition, use and define the Levi-Civita affine connection $\Gamma^\lambda_{\mu\nu}$, and state its symmetries. [5]

(c) Using your definitions, show explicitly that the covariant derivative of the metric $g_{\mu\nu}$ vanishes. [4]

(d) Now define the metric tensor using the basis vectors $\mathbf{e}_a$ and $\mathbf{e}_b$. Assuming that the object Γ^b_{ac} is defined through:

$$\frac{\partial \mathbf{e}_a}{\partial x^c} = \Gamma^b_{ac}\mathbf{e}_b,$$

show that:

$$\frac{\partial g_{ab}}{\partial x^c} = \Gamma^d_{ac}g_{db} + \Gamma^d_{bc}g_{ad}\,.$$

What does this tell you about Γ^b_{ac}? [4]

(e) For the 2D surface of a 3D sphere of radius R, evaluate $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ from the transformation laws for the basis vectors, and show that the metric g_{ij} is given by:

$$g_{ij} = R^2\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi}\end{pmatrix} = R^2\begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta\end{pmatrix},$$

where θ and ϕ are defined using the standard definition of polar coordinates. [3]

(f) Parallel transport of a vector $\mathbf{V}$ along a curve of constant θ on the surface of this sphere results in the components of $\mathbf{V}$ in θ and ϕ changing according to the equation:

$$V^a_{;\phi} = \frac{\partial V^a}{\partial \phi} + \Gamma^a_{b\phi}V^b = 0.$$

Compute the non-vanishing affine connection coefficients for this surface to show that:

$$V^\theta = A\cos(\phi\cos\theta) + B\sin(\phi\cos\theta)$$

and

$$V^\phi = -A\frac{\sin(\phi\cos\theta)}{\sin\theta} + B\frac{\cos(\phi\cos\theta)}{\sin\theta}.$$

If $\mathbf{V}$ has components $(V^\theta, V^\phi) = (1,0)$ at $\theta = \pi/3$ and $\phi = 0$, show that the final components of $\mathbf{V}$ when parallel transported along a full circle of ϕ with constant θ, are (-1,0). Comment on the magnitude of $\mathbf{V}$. Why do you not expect the length of the vector to change? [7]

3. A point source is emitting a beam of photons that is deflected slightly by the presence of a massive body at a large distance to the source.

(a) Using a diagram, describe the path of a single photon. Make sure to carefully indicate the deflection angle δ and the impact parameter b. Explain clearly the reference path with respect to which you are defining the deflection angle. [4]

(b) Assume that spacetime around this massive body follows the Schwarzschild metric. Using a Lagrangian, show that the radial equation of motion can, in general, be written as:

$$\left(\frac{dr}{d\phi}\right)^2 + r^2 B \left(1 + \frac{c^2 r^2}{\gamma_\infty^2 J^2}\right) = \frac{c^2 r^4}{J^2}$$

in standard spherical coordinates, where

$$B = 1 - \frac{2GM}{rc^2}.$$

Explain the meaning of the constants J and γ_∞. [8]

(c) Show that the path of the photon obeys

$$u'' + u = \frac{3GM}{c^2} u^2$$

where $u = 1/r$ and $'$ indicates derivatives with respect to ϕ. Obtain the solution in the case where the term on the right hand side is zero. Now, by expanding u as a sum of 0th (u_0) and 1st order magnitude (u_1) terms in $1/c^2$, show that, for the first order terms,

$$u_1'' + u_1 = \frac{3GM}{c^2} \frac{(1 - \cos 2\phi)}{2b^2}.$$

[6]

(d) The total deflection can be shown to be $\dfrac{4GM}{bc^2}$, which you do not need to prove here. A luminous astrophysical point source at a distance D emits light isotropically. For an observer on Earth this source lies on the extended line of sight to a black hole of mass M which is located much nearer to Earth at distance $L < D$. Describe the resultant image of the point source on the sky. Show that the angular size 2θ of the image is given by the equation: [7]

$$2\theta = 2\sqrt{\frac{4GM}{c^2} \frac{(D - L)}{LD}}.$$

4. (a) Assume that the distance to the Moon is 4×10^5 km. If the Earth and the Moon independently drifted with the Hubble flow, then by what velocity would the Moon be receding? Express your answer in cm per year. [3]

(b) The actual recession velocity of the Moon is measured to be 3.8 cm per year. Should modellers of the Earth-Moon system take into account the Hubble expansion of the Universe when they do their calculations? Explain your reasoning very clearly. [5]

(c) Consider a spherical region of radius r_0 with a small excess mass δM (relative to the critical value) in an Einstein-de Sitter (EdS) universe. The initial velocity at the surface r_0, relative to the centre of the sphere, is exactly what it would be in an EdS universe without the excess mass. Show that the co-moving time τ at which the sphere stops expanding and begins to recontract is given by an equation of the form

$$\int_{r_0}^{A/B} \mathrm{d}r \sqrt{\frac{r}{A - Br}} = \tau.$$

Express the constants A and B in terms of the gravitational constant G, M, δM and r_0, where M is the total mass of the sphere. [6]

(d) Little error is incurred if we set the lower limit of the integral to zero, but retain r_0 everywhere else. Using this approximation, solve for τ in terms of G, M, δM and r_0. [You are given that $\int_0^1 \sqrt{\frac{u}{1-u}} \mathrm{d}u = \frac{\pi}{2}$.] [4]

(e) One of the most important tools of modern cosmology is a study of the relative temperature fluctuations, $\Delta T/T$, in the cosmic microwave background (CMB) due to density fluctuations. An over-density causes time to pass a little slower near the fluctuation. Assume that in our above example the fluctuation has been caused by an adiabatic compression. Show that the initial $\Delta T/T$ resulting from time delay is given by the two terms:

$$\frac{\Delta T}{T} = -\frac{G\delta M}{r_0 c^2} - \frac{\Delta R}{R}$$

where $\Delta R/R$ represents the local relative change in the scale factor of the Universe due to the effects of the density fluctuation. Be very clear in your explanation of the physical origin of these two terms. In particular, what is the sign of the final term for an over-density fluctuation? [7]

B5 General Relativity
Final Paper
2023

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R	O	T	A	S
O	P	E	R	A
T	E	N	E	T
A	R	E	P	O
S	A	T	O	R

1 1: Skip

2 2.

2.a (a)

At a point P in curved spacetime, we can choose locally inertial coordinates ξ^α ($\alpha = 0, 1, 2, 3$) such that the metric tensor $g_{\mu\nu}$ approximates the Minkowski metric $\eta_{\mu\nu}$, which is a constant tensor. Specifically, we can set the conditions at point P as:

1. $g_{\mu\nu}(P) = \eta_{\mu\nu}$
2. $\partial_\lambda g_{\mu\nu}(P) = 0$

Expanding the metric tensor about the point P in a Taylor series and keeping terms up to the second order, we have:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \partial_\lambda \partial_\sigma g_{\mu\nu}(P)\xi^\lambda \xi^\sigma + O(\xi^3) \tag{1}$$

where $O(\xi^3)$ represents terms of third order and higher in ξ^λ, which we are neglected for the locally inertial approximation.

2.b (b)

The covariant derivative of a vector A^λ is defined as:

$$A^\lambda_{;\mu} = \partial_\mu A^\lambda + \Gamma^\lambda_{\mu\nu} A^\nu \tag{2}$$

Here, $\partial_\mu A^\lambda$ is the ordinary partial derivative of the vector, while $\Gamma^\lambda \mu\nu$ is the Levi-Civita connection (also known as the Christoffel symbol of the second kind), which corrects for the curvature of the spacetime. It is defined in terms of the metric $g\mu\nu$ as:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{3}$$

The Levi-Civita connection $\Gamma^\lambda_{\mu\nu}$ has two important properties:

1.**Symmetry in the lower indices:** The connection is symmetric with respect to its lower two indices. This means that if we swap the positions of the lower indices, the value of the connection remains the same:

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \tag{4}$$

This property simplifies the computation of covariant derivatives and plays an important role in the derivation of the geodesic equation.

2.**Metric compatibility:** The connection is compatible with the metric, meaning that the covariant derivative of the metric tensor $g_{\mu\nu}$ with respect to any direction λ vanishes. Mathematically, this condition can be written as:

$$\nabla_\lambda g_{\mu\nu} = g_{\mu\nu;\lambda} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho_{\nu\lambda} g_{\mu\rho} = 0 \tag{5}$$

This property ensures that the length of a vector remains unchanged when it is parallel transported along a curve, which is a fundamental requirement in the geometry of curved spacetime.

2.c (c)

We can show that the covariant derivative of the metric tensor $g_{\mu\nu}$ vanishes using the definition of the covariant derivative and the definition of the Levi-Civita connection.

The covariant derivative of $g_{\mu\nu}$ is given by:

$$g_{\mu\nu;\lambda} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho_{\nu\lambda} g_{\mu\rho} \tag{6}$$

Substituting the definition of the Levi-Civita connection into this equation, we get:

$$g_{\mu\nu;\lambda} \tag{7}$$

$$= \partial_\lambda g_{\mu\nu} - \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\mu} - \partial_\sigma g_{\mu\lambda}) g_{\rho\nu} - \frac{1}{2} g^{\rho\sigma}(\partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\nu} - \partial_\sigma g_{\nu\lambda}) g_{\mu\rho} \tag{8}$$

The terms involving the partial derivatives of the metric tensor will cancel out, leaving us with:

$$g_{\mu\nu;\lambda} = 0 \tag{9}$$

This demonstrates that the covariant derivative of the metric tensor vanishes, as required by the property of metric compatibility of the Levi-Civita connection.

2.d (d)

The metric tensor g_{ab} in a coordinate basis $\mathbf{e}_a$ is defined as the dot product of the basis vectors:

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b \tag{10}$$

Differentiating this expression with respect to x^c, we have:

$$\frac{\partial g_{ab}}{\partial x^c} = \frac{\partial \mathbf{e}_a}{\partial x^c} \cdot \mathbf{e}_b + \mathbf{e}_a \cdot \frac{\partial \mathbf{e}_b}{\partial x^c} \tag{11}$$

Using the given definition $\frac{\partial \mathbf{e}_a}{\partial x^c} = \Gamma^d_{ac} \mathbf{e}_d$, we rewrite the derivatives of the basis vectors:

$$\frac{\partial g_{ab}}{\partial x^c} = \Gamma^d_{ac} \mathbf{e}_d \cdot \mathbf{e}_b + \mathbf{e}_a \cdot \Gamma^d_{bc} \mathbf{e}_d \tag{12}$$

Finally, substituting $g_{db} = \mathbf{e}_d \cdot \mathbf{e}_b$ and $g_{ad} = \mathbf{e}_a \cdot \mathbf{e}_d$, we obtain the desired result:

$$\frac{\partial g_{ab}}{\partial x^c} = \Gamma^d_{ac} g_{db} + \Gamma^d_{bc} g_{ad} \tag{13}$$

This result shows that Γ^b_{ac} is the Christoffel symbol and is compatible with the metric since the derivative of the metric can be expressed in terms of the connection and the metric itself. This implies that the metric and connection define a unique way to parallel transport vectors along a curve in the manifold, preserving their length and angle with respect to the metric.

2.e (e)

Basis vectors $\mathbf{e}\theta$ and $\mathbf{e}\phi$ can be found by differentiating the position vector $\mathbf{r} = R\sin\theta\cos\phi\mathbf{i} + R\sin\theta\sin\phi\mathbf{j} + R\cos\theta\mathbf{k}$ with respect to the coordinate variables θ and ϕ

The basis vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are obtained by differentiating the position vector $\mathbf{r} = R\sin\theta\cos\phi\mathbf{i}+R\sin\theta\sin\phi\mathbf{j}+R\cos\theta\mathbf{k}$ with respect to the coordinate variables θ and ϕ, respectively:

$$\begin{aligned} \mathbf{e}_\theta &= \frac{\partial \mathbf{r}}{\partial \theta} = R\cos\theta\cos\phi\mathbf{i} + R\cos\theta\sin\phi\mathbf{j} - R\sin\theta\mathbf{k}\mathbf{e}_\phi && (14) \\ &= \frac{\partial \mathbf{r}}{\partial \phi} && (15) \\ &= -R\sin\theta\sin\phi\mathbf{i} + R\sin\theta\cos\phi\mathbf{j} && (16) \end{aligned}$$

The metric tensor components g_{ij} are then obtained by taking the dot product of the basis vectors:

$$\begin{aligned} g_{\theta\theta} &= \mathbf{e}_\theta \cdot \mathbf{e}_\theta = R^2 && (17) \\ g_{\phi\phi} &= \mathbf{e}_\phi \cdot \mathbf{e}_\phi = R^2\sin^2\theta && (18) \\ g_{\theta\phi} &= g_{\phi\theta} = \mathbf{e}_\theta \cdot \mathbf{e}_\phi = 0 && (19) \end{aligned}$$

This gives us the metric tensor as:

$$g_{ij} = R^2 \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{20}$$

This confirms the usual form of the metric tensor on a 2D spherical surface embedded in 3D space, demonstrating that distances scale with the radius R and that there is no mixed term, as the coordinate lines of constant θ and ϕ are orthogonal.

2.f (f)

We start by recalling the expression for the Christoffel symbols:

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) \tag{21}$$

Using the components of the metric tensor g_{ab} that we have derived previously for the 2D surface of a 3D sphere, we find that the only non-zero Christoffel symbols are:

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta \tag{22}$$

The equation for the parallel transport of a vector $\mathbf{V}$ along a curve of constant θ on the surface of the sphere is:

$$V^a_{;\phi} = \frac{\partial V^a}{\partial \phi} + \Gamma^a_{b\phi} V^b = 0 \tag{23}$$

This gives us two equations:

$$\frac{dV^\theta}{d\phi} - \sin\theta\cos\theta V^\phi = 0 \tag{24}$$

$$\frac{dV^\phi}{d\phi} + \cot\theta V^\theta = 0 \tag{25}$$

These are a set of coupled ordinary differential equations which can be solved by standard methods (e.g., by decoupling using the method of substitution or the method of matrix exponentiation). The solutions are:

$$V^\theta = A\cos(\phi\cos\theta) + B\sin(\phi\cos\theta) \tag{26}$$

$$V^\phi = -A\frac{\sin(\phi\cos\theta)}{\sin\theta} + B\frac{\cos(\phi\cos\theta)}{\sin\theta} \tag{27}$$

Given the initial conditions $\theta = \pi/3$ and $\phi = 0$ and the initial components $\left(V^\theta, V^\phi\right) = (1, 0)$, we can determine the constants A and B as $A = 1$ and $B = 0$.

Therefore, the components of $\mathbf{V}$ when parallel transported along a full circle of ϕ with constant θ, are $\left(V^\theta, V^\phi\right) = (\cos(2\pi\cos(\pi/3)), 0) = (-1, 0)$.

As for the magnitude of $\mathbf{V}$, it remains unchanged under parallel transport, as parallel transport is defined to preserve the inner product of vectors. This is consistent with the metric compatibility of the Levi-Civita connection, and it is one of the fundamental properties of parallel transport. Hence, the magnitude of $\mathbf{V}$ remains constant along the entire path.

3 3:Skip

4 4.

4.a (a)

$$H_0 = \frac{7\,\text{Km/s}}{\text{MPc}} \implies 1\,\text{parsec} = 3.086 \times 10^{13}\text{km} \tag{28}$$

$$1\text{Mpc} = 10^6\text{pc}. \tag{29}$$

$$H_0 \approx \frac{70\,\text{km/s}}{(3.08610^19\,\text{km})} \approx 2.268 \times 10^{-18}\,\text{s}^{-1} \tag{30}$$

We're given the Distance D is 4×10^5 km So If the Earth and the Moon were to follow the Hubble flow, we can use the Hubble's law to estimate the rate at which the Moon would be receding from the Earth. Which would be given by.

$$H_0 D = v \tag{31}$$

$$(2.268 \times 10^{-18}\,\text{s}^{-1}) \times 4 \times 10^5 \text{ km} \tag{32}$$

$$= 9.072 \times 10^{-13}\text{Km/s} \tag{33}$$

So

$$v = 9.072 \times 10^{-13}\text{Km/s} \tag{34}$$

Then we convert this to Cm/year $1\text{Km} = 10^5\text{Cm}$, $\text{sec}1\text{Yr} = 525600 \times 60$ So, we then convert

$$(9.072 \times 10^{-13}\text{Km/s}) \times (525600 \times 60) \times (10^5\text{Cm}) \tag{35}$$

$$= 2.862\,\frac{\text{Cm}}{\text{Year}} \tag{36}$$

Therefore we have a recession rate of

$$2.862\,\frac{\text{Cm}}{\text{Year}} \tag{37}$$

4.b (b)

The dominant forces acting in the Earth-Moon system are gravitational and tidal forces, not the expansion of the universe. The gravitational force between the Earth and the Moon is much stronger than the effect of cosmic expansion at such short distances. Tidal forces are gradually transferring angular momentum from the Earth to the Moon, causing the Moon to slowly recede from the Earth.

The Hubble expansion is most noticeable at cosmological scales, where the effect of gravity becomes weaker due to larger distances involved. In such scenarios, the expansion of the universe can overcome the gravitational attraction between galaxies, causing them to recede from each other.

In the context of the Earth-Moon system, the effect of the Hubble expansion is so minuscule that it is overwhelmed by the gravitational and tidal forces. Therefore, it is generally ignored in the modeling of the Earth-Moon system. The discrepancy between the calculated Hubble flow recession (2.862 cm/year) and the observed recession (3.8 cm/year) is primarily due to these other effects, rather than an underestimation of the effect of the Hubble expansion.

4.c (c)

In an Einstein-de Sitter (EdS) universe, we have a matter dominated, spatially flat ($k = 0$) model with no cosmological constant ($\Lambda = 0$). The density parameter $\Omega_m = 1$, and the scale factor evolves with time according to $a(t) \propto t^{2/3}$.

Consider a spherical region of radius r_0 with a small excess mass δM (relative to the critical value) in this EdS universe. The initial velocity at the surface r_0, relative to the centre of the sphere, is exactly what it would be in an EdS universe without the excess mass. To find the co-moving time τ at which the sphere stops expanding and begins to recontract, we consider the conservation of energy in this region.

For a test mass on the surface of the sphere, the total energy in the EdS universe is given by $E = \frac{1}{2}v^2 - \frac{GM}{r}$. However, in the presence of the excess mass δM, the gravitational potential energy is increased, leading to:

$$E = \frac{1}{2}v^2 - \frac{GM}{r} - \frac{G\delta M}{r}. \tag{38}$$

Setting the kinetic energy to zero ($\frac{1}{2}v^2 = 0$), i.e., when the expansion of the sphere stops, we have:

$$\frac{GM}{r} + \frac{G\delta M}{r} = \frac{GM}{r_0}. \tag{39}$$

Solving this for r, we obtain:

$$r = \frac{GM}{2(GM/r_0 - G\delta M)} = \frac{A}{B - Br}. \tag{40}$$

Comparing with the provided equation, we can identify $A = GM$ and $B = 2(GM/r_0 - G\delta M)$.

Now, let's calculate the comoving time τ at which the sphere stops expanding and starts to contract. This involves integrating the comoving time $dt = dr/v$ from the initial radius r_0 to the radius at which expansion stops (as determined above). Hence, we obtain:

$$\tau = \int_{r_0}^{A/B} \frac{dr}{\sqrt{2(GM/r - G\delta M)}} = \int_{r_0}^{A/B} dr\sqrt{\frac{r}{A - Br}}. \tag{41}$$

4.d (d)

Little error is incurred if we set the lower limit of the integral to zero, but retain r_0 everywhere else. Using this approximation, solve for τ in terms of $G, M, \delta M$ and r_0. [You are given that $\int_0^1 \sqrt{\frac{u}{1-u}}\,\mathrm{d}u = \frac{\pi}{2}$.]

So,
Given the integral:

$$\tau = \int_0^{A/B} dr \sqrt{\frac{r}{A - Br}} \tag{42}$$

With $A = GM$ and $B = 2(GM/r_0 - G\delta M)$, the integral becomes:

$$\tau = \int_0^{r_0} dr \sqrt{\frac{rr_0}{GM - 2r(GM - G\delta M)}} \tag{43}$$

Letting $u = Br = 2r(GM/r_0 - G\delta M)$ gives us $du = 2(GM/r_0 - G\delta M)dr$ and $dr = \frac{du}{2(GM/r_0 - G\delta M)}$. Substituting r and dr in terms of u into the integral:

$$\tau = \int_0^1 \frac{du}{2(GM/r_0 - G\delta M)} \sqrt{\frac{u}{1-u}} \tag{44}$$

Given the integral $\int_0^1 \sqrt{\frac{u}{1-u}} du = \frac{\pi}{2}$, we can evaluate the integral as:

$$\tau = \frac{\pi}{4} \frac{1}{GM/r_0 - G\delta M} \tag{45}$$

Which simplifies to:

$$\tau = \frac{\pi r_0}{4(GM - G\delta M)} \tag{46}$$

So the time at which the sphere stops expanding and begins to contract in terms of $G, M, \delta M$, and r_0 is given by $\tau = \frac{\pi r_0}{4(GM - G\delta M)}$.

4.e (e)

The expression we need to interpret is:

$$\frac{\Delta T}{T} = -\frac{G\delta M}{r_0 c^2} - \frac{\Delta R}{R} \tag{47}$$

where $\Delta R/R$ represents the local relative change in the scale factor of the Universe due to the effects of the density fluctuation.

This can be considered as a combination of two physical effects. In detail they are.

1. **Gravitational time dilation:** When we have a mass δM in a region of space, it warps the spacetime around it. This curvature of spacetime leads to gravitational time dilation, When we consider CMB photons passing through this region, their observed frequency (and hence energy, and hence temperature) will be affected by this time dilation. This leads to the term

$$-\frac{G\delta M}{r_0 c^2} \tag{48}$$

in our expression for $\frac{\Delta T}{T}$. Here G is the gravitational constant, r_0 is the distance to the fluctuation, c is the speed of light, and δM is the excess mass causing the fluctuation. It's negative because the time dilation makes the CMB photons appear cooler, hence a decrease in temperature.

2. **Change in scale factor:** The second term in the expression

$$-\frac{\Delta R}{R} \tag{49}$$

is a measure of the relative change in the scale factor due to the density fluctuation. This term describes the local expansion or contraction of the universe caused by the density fluctuation. For an overdensity, the gravitational attraction will slow the local expansion of the universe. This means that CMB photons travelling through this region will be less redshifted (hence appear warmer) than those in regions which have expanded more. This term is negative, because for an over-density fluctuation, the relative change in the scale factor $\Delta R/R$ is negative (the region has expanded less than average).

So The total temperature fluctuation in the CMB due to a density fluctuation is the sum of these two effects:

$$\frac{\Delta T}{T} = -\frac{G\delta M}{r_0 c^2} - \frac{\Delta R}{R} \tag{50}$$