

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part A

A1: THERMAL PHYSICS

TRINITY TERM 2022

Monday, 13 June, 2.30 pm – 5.30 pm

*Answer **all** of Section A and **three** questions from Section B.*

*For Section A start the answer to each question on a **fresh page**.*

*For Section B start the answer to each question in a **fresh book**.*

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Explain what is meant by *function of state*. A body is heated and its temperature increases from T_i to T_f . Write down an expression, in terms of the relevant quantities, for the change in its entropy. [4]

2. Derive the Maxwell relation: $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$. [2]

Calculate the total entropy changes when:

(i) a mole of ideal gas undergoes a Joule expansion from volume V to $2V$. [2]

(ii) two different gases (a mole of each) are initially in separate halves of a thermally isolated box of total volume $2V$, and then the barrier between the two parts of the box is removed allowing the gases to mix. [2]

3. State briefly the physical origin of the terms a/V^2 and b in the van der Waals equation of state:

$$p = \frac{RT}{V-b} - \frac{a}{V^2},$$

where R is the molar gas constant. [3]

Determine the work done by the gas during an isothermal expansion of one mole of gas from an initial volume V_1 to a final volume V_2 . Determine the change in the internal energy U of the gas during this process and comment on its sign. [4]

4. The equilibrium separation between hydrogen nuclei in the hydrogen molecule is 0.074 nm and the force constant of the bond is 580 Nm^{-1} . Estimate the temperature needed to excite (i) the lowest rotational mode and (ii) the lowest vibrational mode. [6]

5. A car tyre is inflated to pressure 270 kPa at the beginning of a trip. After the journey the pressure in the tyre is found to be 300 kPa . What is the internal energy change of the air in the tyre between the pressure measurements? You may assume that the air is an ideal gas with a constant molar heat capacity $C_V = 5R/2$ and that the internal volume of the tyre remains constant at $5.7 \times 10^{-3}\text{ m}^3$. [5]

6. An ensemble of one-dimensional harmonic oscillators with natural angular frequency $\omega = 6 \times 10^6\text{ rad s}^{-1}$ is prepared with every member in its 3rd excited state (i.e. vibrational quantum number $n = 3$) and then isolated as the ensemble relaxes to thermal equilibrium. Find the final temperature. [7]

7. (a) A small spherical satellite with uniform temperature is in orbit around the Sun. The Sun subtends a solid angle of 7×10^{-5} steradians at the satellite. Assuming that the emissivity of the satellite is independent of wavelength and that the temperature of the Sun's surface is 5780 K, calculate the temperature of the satellite. [4]

(b) It is proposed to coat the satellite with a layer of paint so as to lower its temperature. Sketch a suitable curve of emissivity of the paint, as a function of wavelength, so as to achieve this. [1]

Section B

8. (a) Consider an ideal gas having constant heat capacity C_V . Find the relationship between pressure p , energy density u and adiabatic index γ . Explain briefly which parts of your derivation would change if the gas were not ideal. [5]

(b) Find an expression for the pressure gradient in a fluid of density $\rho(r)$ in equilibrium in a gravitational field \mathbf{g} (hydrostatic equilibrium). [2]

(c) Under what conditions is it the case that

$$\int_{x_1}^{x_2} x^3 \frac{df}{dx} dx = -3 \int_{x_1}^{x_2} x^2 f dx$$

where $f(x)$ is some general function of x ? [2]

(d) Consider a hot spherical cloud of gravitating particles in otherwise empty space (i.e. like a star, but without any nuclear reactions). This can be modelled by treating each small part as an ideal gas. The pressure, temperature and density ρ are all functions of r . Let $M(r)$ be the mass of the material within a radius r :

$$M(r) = \int_0^r \rho 4\pi r'^2 dr'.$$

Write down the relationship between the pressure gradient dp/dr and properties of the cloud in hydrostatic equilibrium. The pressure goes to zero at the surface of the cloud: explain briefly why this must be so. Hence show that the total thermal energy, U , and the total gravitational potential energy, Ω , are related by

$$3(\gamma - 1)U + \Omega = 0. \quad [6]$$

(e) A cloud with N particles and $\gamma = 5/3$ radiates away some energy ΔQ . Find the change in temperature of the cloud, paying attention to the sign. Hence find the heat capacity of the cloud. If two such clouds are exchanging radiant heat, describe qualitatively what will happen over time. [5]

9. (a) Show that if $I_n = \int_0^\infty x^n \exp(-\alpha x^2) dx$ then $2\alpha I_n = (n-1)I_{n-2}$. [2]

A monatomic gas is held at low pressure p and temperature T . The velocity distribution function has the form:

$$C \exp(-\alpha[v_x^2 + v_y^2 + v_z^2])$$

(b) Write down an expression for α in terms of the relevant thermodynamics quantities. [1]

(c) Derive an expression for the energy distribution function and sketch the distribution. [5]

(d) Calculate the mean kinetic energy per atom in the gas at temperature T . [2]

(e) A container of gas at temperature T has a hole whose radius is small compared to the mean free path. Gas particles stream through the hole into a vacuum. A box is opened for a short time and catches some of the particles. Find the final temperature of the gas trapped in the box in the cases: (i) a monatomic gas, (ii) a diatomic gas in which rotation is fully thermally excited and vibration is not excited. (In all cases the thermal capacity of the box can be ignored.) [7]

Explain qualitatively what you would expect in a similar experiment with a hole of radius larger than the mean free path (but still small compared to other distances in the apparatus). [3]

10. A paramagnetic solid has n atoms per unit volume, each with a permanent magnetic dipole moment μ . In an applied magnetic field, B , each atom has two energy eigenstates with energies $\pm\mu B$.

(a) Obtain an expression for the magnetic susceptibility of the solid in terms of the above properties and temperature T , assuming n is small and the atoms behave independently. [5]

(b) Obtain the relationship between entropy S and total dipole moment m of the solid. [3]

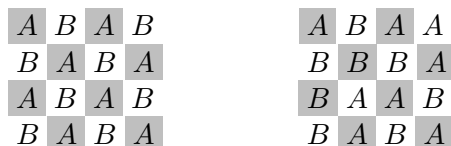
(c) Let N be the total number of dipoles. Sketch the form of the magnetic heat capacity C as a function of temperature, including labelled estimates of the main features (the full expression is not required), and state the value of $\int_0^\infty (C/T) dT$. [4]

(d) The solid is prepared at temperature 1 K in a magnetic field of 1 T. It is then thermally isolated and the field is reduced adiabatically to 0.3 T. Describe what happens to the dipoles, and find the final temperature. [3]

(e) Outline qualitatively what happens in a real (not idealized) experiment in the following two cases:

- (i) the field is reduced adiabatically to 0 T;
- (ii) the direction of the field is abruptly reversed. [5]

11. In a simple model of a binary alloy there are two types of atom, A and B , and two types of lattice site, 1 and 2. In a given crystal there are N sites of each type and N atoms of each type. In a state of complete order, all the A atoms are on sites of type 1, and all the B atoms are on sites of type 2, as in the left diagram below which shows an example with $N = 8$ and type 1 sites shaded. At non-zero temperature some atoms swap sites, as illustrated in the right diagram.



A microstate can be specified by stating the type of atom at each site. A macrostate can be specified by giving the values of N and n . A state in which $2n$ atoms (n of each type) have moved has energy $U = n\epsilon$.

(a) For a given N , write down an expression for the number of microstates having energy $n\epsilon$. Hence find the entropy of this macrostate, using the microcanonical ensemble, and show that in the case where $N, n \gg 1$ the result can be written approximately as

$$S = -a(p_1 \ln p_1 + p_2 \ln p_2)$$

where you should give expressions for the constant a and the variables p_1 and p_2 . [4]

(b) Throughout the rest of the question, assume $N, n \gg 1$. Find the temperature of the equilibrium state of energy $U = n\epsilon$ in terms of N, n, ϵ and k_B . Simplify your expression for the cases (i) $n \ll N$ and (ii) $n = N/2$ and comment. [5]

(c) Now consider a single site. Let the energy of the site be 0 when it is occupied by an atom of one type, and $\epsilon/2$ when it is occupied by an atom of the other type. Confirm that this assignment is consistent with $U = n\epsilon$ for the whole crystal. A single site can be treated in the canonical ensemble by regarding the rest of the crystal as a thermal reservoir. Adopting this approach, write down the partition function for a single site, and hence obtain the mean energy \bar{U}_1 of a single site in conditions of thermal equilibrium. Confirm that your expression is consistent with the result of part (b). [5]

(d) Assume that each atom interacts only with its nearest neighbours through an attractive energy of $-E$ ($E > 0$) between like neighbours A-A and B-B, and a repulsive energy of $+E$ for an A-B pair. What is the state of the system at zero temperature? [3]

(e) Estimate the total interaction energy assuming that the atoms are randomly distributed among the N sites; i.e. each site is occupied independently with probabilities $p_A = N_A/N$ and $p_B = N_B/N$. [3]

A1: Thermal And Statistical Physics
Final Paper TT 2023
Practice Paper from the 2022 Paper
Taken Under Test Conditions 2 Hr 30 Mins

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R O T A S
O P E R A
T E N E T
A R E P O
S A T O R

A Section A

A.1 1.

A state function, also known as a thermodynamic property, is a property of a system that depends only on the current state of the system, and is independent of the process taken to reach that state. Examples include pressure (P), volume (V), temperature (T), internal energy (U), enthalpy (H), and entropy (S).

The change in entropy (ΔS) of a body heated from an initial temperature T_i to a final temperature T_f is given by:

$$\Delta S = \int_{T_i}^{T_f} \frac{C(T)}{T} dT, \quad (\text{A.1.1})$$

where $C(T)$ is the relevant heat capacity. This could be the heat capacity at constant pressure $C_p(T)$ or at constant volume $C_v(T)$, depending on the conditions of the process. If the heat capacity is approximately constant over the temperature range of interest, the equation simplifies to:

$$\Delta S = C \ln \frac{T_f}{T_i}. \quad (\text{A.1.2})$$

A.2 2.

We start with the thermodynamic potential $F = U - TS$, where U is the internal energy, T is the temperature, S is the entropy, and F is the Helmholtz free energy. The total differential of F is:

$$dF = -SdT - PdV, \quad (\text{A.2.1})$$

where P is the pressure and V is the volume. The desired Maxwell relation can be derived by calculating the mixed second derivatives, which should be equal due to Schwarz's theorem:

$$\left(\frac{\partial^2 F}{\partial T \partial V} \right) = \left(\frac{\partial^2 F}{\partial V \partial T} \right). \quad (\text{A.2.2})$$

Performing these derivatives on the left and right side, we get the Maxwell relation:

$$\left(\frac{\partial S}{\partial V} \right)_T = \left(\frac{\partial P}{\partial T} \right)_V. \quad (\text{A.2.3})$$

Now, let's calculate the entropy changes in the two given situations:

(i) Consider an ideal gas undergoing a Joule expansion from volume V to $2V$ at constant temperature. The entropy change ΔS in this isothermal process can be calculated from the Sackur-Tetrode equation for the entropy of an ideal gas:

$$S = Nk \left[\ln \left(\frac{V}{N} \left(\frac{4\pi mU}{3Nh^2} \right)^{3/2} \right) + \frac{5}{2} \right]. \quad (\text{A.2.4})$$

Neglecting terms that do not depend on V , we can write the entropy difference as

$$\Delta S = Nk \ln \frac{V_f}{V_i} = R \ln 2. \quad (\text{A.2.5})$$

(ii) Consider two different gases (a mole of each) initially in separate halves of a thermally isolated box of total volume $2V$. The entropy change when the barrier is removed and the gases mix consists of the entropy changes for each gas due to expansion plus the entropy change due to the mixing.

The expansion entropy change for each gas is $R \ln 2$, so the total expansion entropy change is $2R \ln 2$.

The mixing entropy change for each gas can be calculated using the formula for entropy change upon mixing of two ideal gases:

$$\Delta S_{\text{mix}} = -nR \sum_i x_i \ln x_i, \quad (\text{A.2.6})$$

where $x_i = 0.5$ is the mole fraction of each gas. This yields $\Delta S_{\text{mix}} = R \ln 2$. So, the total entropy change is $\Delta S_{\text{total}} = \Delta S_{\text{total,expansion}} + \Delta S_{\text{mix}} = 3R \ln 2$.

A.3 3.

The van der Waals equation of state, given by

$$p = \frac{RT}{V-b} - \frac{a}{V^2}, \quad (\text{A.3.1})$$

accounts for the non-ideal behavior of gases. Here, a and b are the van der Waals constants, which account for intermolecular attractions and molecular volume, respectively. The work done W during an isothermal expansion of one mole of gas from an initial volume V_1 to a final volume V_2 can be calculated by integrating the pressure p with respect to volume V : The work done W during an isothermal expansion of one mole of gas from an initial volume V_1 to a final volume V_2 can be calculated by integrating the pressure p with respect to volume V :

$$W = - \int_{V_1}^{V_2} p dV = - \int_{V_1}^{V_2} \left(\frac{RT}{V-b} - \frac{a}{V^2} \right) dV. \quad (\text{A.3.2})$$

This integral can be split into two parts:

$$W = - \int_{V_1}^{V_2} \frac{RT}{V-b} dV + \int_{V_1}^{V_2} \frac{a}{V^2} dV. \quad (\text{A.3.3})$$

The first integral can be solved by substitution. Let $u = V-b$, then $dV = du$ and when $V = V_1$, $u = V_1 - b$, and when $V = V_2$, $u = V_2 - b$. Substituting these values gives:

$$- \int_{V_1}^{V_2} \frac{RT}{V-b} dV \quad (\text{A.3.4})$$

$$= -RT \int_{V_1-b}^{V_2-b} \frac{1}{u} du \quad (\text{A.3.5})$$

$$= -RT \ln u \Big|_{V_1-b}^{V_2-b} \quad (\text{A.3.6})$$

$$= -RT \ln \left(\frac{V_2-b}{V_1-b} \right). \quad (\text{A.3.7})$$

The second integral is a standard integral of the form $\int x^{-n} dx = \frac{x^{1-n}}{1-n}$, which gives:

$$\int_{V_1}^{V_2} \frac{a}{V^2} dV \quad (\text{A.3.8})$$

$$= a \int_{V_1}^{V_2} V^{-2} dV \quad (\text{A.3.9})$$

$$= a \left[-\frac{1}{V} \right]_{V_1}^{V_2} \quad (\text{A.3.10})$$

$$= a \left(\frac{1}{V_1} - \frac{1}{V_2} \right). \quad (\text{A.3.11})$$

Therefore, the total work done during the isothermal expansion is:

$$W = -RT \ln \left(\frac{V_2 - b}{V_1 - b} \right) + a \left(\frac{1}{V_1} - \frac{1}{V_2} \right). \quad (\text{A.3.12})$$

For an isothermal process, the change in internal energy ΔU of the system is zero due to the first law of thermodynamics:

$$\Delta U = Q - W = 0, \quad (\text{A.3.13})$$

where Q is the heat absorbed by the system. This implies that the work done by the system is compensated by the heat absorbed from the surroundings.

A.4 4.

We are given the equilibrium separation between hydrogen nuclei in the hydrogen molecule as 0.074 nm and the force constant of the bond as 580 N/m. Our goal is to estimate the temperature needed to excite (i) the lowest rotational mode and (ii) the lowest vibrational mode.

(i) To find the temperature needed to excite the lowest rotational mode, we first need to calculate the moment of inertia I of the hydrogen molecule. The moment of inertia can be expressed as:

$$I = \mu R^2, \quad (\text{A.4.1})$$

where μ is the reduced mass of the hydrogen molecule and R is the equilibrium separation between hydrogen nuclei. The reduced mass can be calculated as:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad (\text{A.4.2})$$

where m_1 and m_2 are the masses of the hydrogen nuclei. Since both nuclei are hydrogen, their masses are equal, and we can simplify the reduced mass as:

$$\mu = \frac{m}{2}, \quad (\text{A.4.3})$$

where m is the mass of a hydrogen nucleus. Now, we can calculate the moment of inertia:

$$I = \frac{m}{2}(0.074 \times 10^{-9})^2. \quad (\text{A.4.4})$$

The energy of the lowest rotational mode can be found using the following formula:

$$E_{\text{rot}} = \frac{\hbar^2}{2I}, \quad (\text{A.4.5})$$

where \hbar is the reduced Planck constant. To excite the lowest rotational mode, we need to supply this energy in the form of thermal energy, which can be expressed as:

$$k_B T_{\text{rot}} \approx E_{\text{rot}}, \quad (\text{A.4.6})$$

where k_B is the Boltzmann constant. Solving for the temperature T_{rot} , we get:

$$T_{\text{rot}} \approx \frac{\hbar^2}{2Ik_B}. \quad (\text{A.4.7})$$

Plugging in the values, we find that $T_{\text{rot}} \approx 88.08$ K.

(ii) To find the temperature needed to excite the lowest vibrational mode, we first need to calculate the vibrational frequency ω of the hydrogen molecule using the force constant k :

$$\omega = \sqrt{\frac{k}{\mu}}. \quad (\text{A.4.8})$$

The energy of the lowest vibrational mode can be found using the following formula:

$$E_{\text{vib}} = \hbar\omega. \quad (\text{A.4.9})$$

To excite the lowest vibrational mode, we need to supply this energy in the form of thermal energy, which can be expressed as:

$$k_B T_{\text{vib}} \approx E_{\text{vib}}, \quad (\text{A.4.10})$$

where k_B is the Boltzmann constant. Solving for the temperature T_{vib} , we get:

$$T_{\text{vib}} \approx \frac{\hbar\omega}{k_B}. \quad (\text{A.4.11})$$

Plugging in the values, we find that $T_{\text{vib}} \approx 6365.96$ K.

A.5 5.

Given the initial and final pressures P_1 and P_2 , the constant volume V , and the molar heat capacity at constant volume $C_V = \frac{5R}{2}$, we want to find the change in internal energy ΔU of the air in the tyre.

Given $PV = nRT$,

Let,

$$n = \frac{P_1 V}{RT}.$$

Assume process is isothermal.

$$\Delta U = \Delta W = nRT \ln \left(\frac{P_2}{P_1} \right).$$

Substituting the given values

$$P_1 = 270 \text{ kPa} \tag{A.5.1}$$

$$P_2 = 300 \text{ kPa} \tag{A.5.2}$$

$$V = 5.7 \times 10^{-3} \text{ m}^3 \tag{A.5.3}$$

$$C_V = \frac{5R}{2} \tag{A.5.4}$$

We convert to pascals by multiplying by 10^3 ,

Let $T = 1K$

$$\Delta U = \Delta W = \frac{270 \times 10^3 \times 5.7 \times 10^{-3}}{8.314} T \ln \left(\frac{300 \times 10^3}{270 \times 10^3} \right) \tag{A.5.5}$$

$$= 185.24 \times T \times \ln \left(\frac{300}{270} \right) \approx 19.5 \text{ J} \tag{A.5.6}$$

A.6 6.

$$\hbar = 1.054 \times 10^{-34} \text{ J s} \quad (\text{A.6.1})$$

$$\omega = 6 \times 10^6 \text{ rad s}^{-1} \quad (\text{A.6.2})$$

$$k_B = 1.381 \times 10^{-23} \text{ J/K} \quad (\text{A.6.3})$$

$$n = 3 \quad (\text{A.6.4})$$

The expression for the temperature is:

$$T = \frac{\hbar\omega \left(n + \frac{1}{2}\right)}{k_B} \quad (\text{A.6.5})$$

Substituting the given values, we get:

$$T = \frac{(1.054 \times 10^{-34} \text{ J s})(6 \times 10^6 \text{ rad s}^{-1}) \left(3 + \frac{1}{2}\right)}{1.381 \times 10^{-23} \text{ J/K}} \quad (\text{A.6.6})$$

$$= \frac{(1.054 \times 10^{-34} \text{ J s})(6 \times 10^6 \text{ rad s}^{-1})(3.5)}{1.381 \times 10^{-23} \text{ J/K}} \quad (\text{A.6.7})$$

$$= \frac{2.2091 \times 10^{-28} \text{ J}}{1.381 \times 10^{-23} \text{ J/K}} \quad (\text{A.6.8})$$

$$= 1.6 \times 10^4 \text{ K} \quad (\text{A.6.9})$$

A.7 7.

A.7.a (a)

Given:

- Solid angle subtended by the Sun, $\Omega = 7 \times 10^{-5}$ steradians,
- Temperature of the Sun, $T_s = 5780$ K,
- Emissivity of the satellite, $\epsilon \approx 1$,

We use the equation:

$$T_{\text{sat}} = \left(\frac{\Omega}{4\pi\epsilon} T_s^4 \right)^{1/4} \quad (\text{A.7.1})$$

Plugging in the given values, we get:

$$T_{\text{sat}} = \left(\frac{7 \times 10^{-5}}{4\pi \times 1} \times (5780)^4 \right)^{1/4} \quad (\text{A.7.2})$$

$$= (7 \times 10^{-5} \times 5.67 \times 10^{-8} \times (5780)^4 / 4\pi)^{1/4} \quad (\text{A.7.3})$$

$$\approx 285 \text{ K.} \quad (\text{A.7.4})$$

Thus, the temperature of the satellite is approximately 285 K.

A.7.b (b)

To lower the temperature of the satellite, we need to choose a paint with an emissivity that is high in the infrared region (where the satellite radiates most of its energy) and low in the visible region (where the Sun emits most of its energy). This will maximize the amount of heat the satellite radiates while minimizing the amount it absorbs from the Sun.

A suitable curve of emissivity as a function of wavelength would be a curve that has a low value in the visible region (approximately 400 – 700 nm) and a high value in the infrared region (wavelengths greater than 700 nm). The curve should have a sharp increase in emissivity starting at around 700 nm and maintain a high value for longer wavelengths. This will ensure that the paint reflects most of the solar radiation while efficiently emitting the satellite's thermal radiation.

In essence, we're looking for a coating that can effectively act as a thermal mirror in the visible spectrum and a thermal radiator in the infrared spectrum.

Emissivity of the proposed paint as a function of wavelength

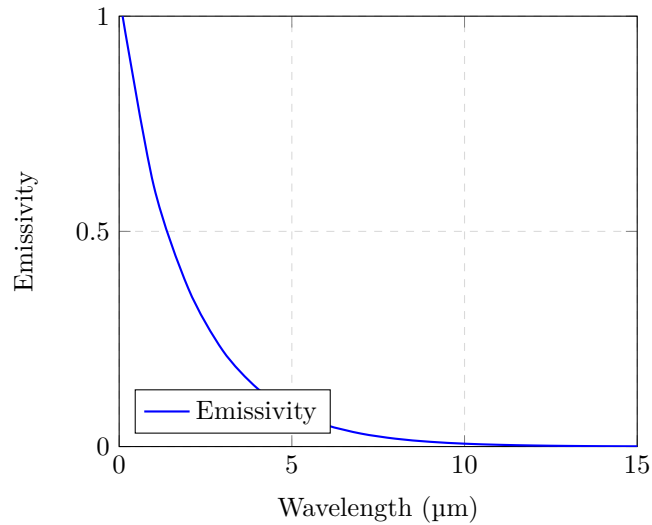


Figure 1: A suitable curve of emissivity as a function of wavelength for the paint.

In this sketch, the emissivity is low (around 0.1) for wavelengths in the visible region, and then sharply increases to a high value (around 0.9) for wavelengths in the infrared region. By using this paint, the satellite will absorb less solar radiation in the visible range and emit more of its own thermal radiation in the infrared range, thus lowering its temperature.

B Section B

B.8 8.

B.8.a (a)

Consider an ideal gas with constant heat capacity C_V . Recall that for an ideal gas, the internal energy U is given by:

$$U = \frac{3}{2}nRT \quad (\text{B.8.1})$$

where n is the number of moles, R is the gas constant, and T is the temperature. The energy density u is the internal energy per unit volume, so:

$$u = \frac{U}{V} = \frac{3}{2} \frac{nRT}{V} \quad (\text{B.8.2})$$

From the ideal gas law, we have:

$$pV = nRT \quad (\text{B.8.3})$$

Dividing both sides by V , we get:

$$p = \frac{nRT}{V} \quad (\text{B.8.4})$$

Now we can express the energy density u in terms of pressure p :

$$u = \frac{3}{2}p \quad (\text{B.8.5})$$

The adiabatic index γ is defined as:

$$\gamma = \frac{C_p}{C_V} \quad (\text{B.8.6})$$

For an ideal gas, the heat capacities are related by:

$$C_p - C_V = R \quad (\text{B.8.7})$$

So we can express γ in terms of C_V :

$$\gamma = \frac{C_V + R}{C_V} \quad (\text{B.8.8})$$

If the gas were not ideal, the relationship between the internal energy and the temperature, as well as the ideal gas law, would be modified. This would affect the expression for the energy density and the relationship between pressure and energy density.

B.8.b (b)

For a fluid of density $\rho(\mathbf{r})$ in equilibrium in a gravitational field \mathbf{g} , the force balance (hydrostatic equilibrium) requires that the pressure gradient balance the gravitational force acting on the fluid. This can be expressed as:

$$\nabla p = -\rho(\mathbf{r})\mathbf{g} \quad (\text{B.8.9})$$

B.8.c (c)

The given integral condition is:

$$\int_{x_1}^{x_2} x^3 \frac{df}{dx} dx = -3 \int_{x_1}^{x_2} x^2 f dx \quad (\text{B.8.10})$$

This condition is satisfied when $f(x)$ is an odd function, i.e., $f(-x) = -f(x)$. To show this, consider the substitution $u = x^3$. Then, $du = 3x^2 dx$, and the given condition becomes:

$$\int_{u(x_1)}^{u(x_2)} \frac{df}{du} du = - \int_{u(x_1)}^{u(x_2)} f(u) du \quad (\text{B.8.11})$$

If $f(u)$ is an odd function, the integrals on both sides of the equation will be equal and opposite, satisfying the given condition.

B.8.d (d)

In a state of hydrostatic equilibrium, the pressure gradient is balanced by the gravitational force. This can be expressed as:

$$\frac{dp}{dr} = -\frac{GM(r)\rho(r)}{r^2} \quad (\text{B.8.12})$$

where G is the gravitational constant, $M(r)$ is the mass enclosed within a radius r , and $\rho(r)$ is the density of the cloud at radius r .

The pressure must go to zero at the surface of the cloud ($r = R$), because there is no overlying material to create a pressure gradient. This is a boundary condition for the above differential equation.

The total thermal energy U of the cloud can be expressed as:

$$U = \int_0^R u 4\pi r^2 dr \quad (\text{B.8.13})$$

where u is the energy density. For an ideal gas, $u = \frac{3}{2}p$, so:

$$U = \frac{3}{2} \int_0^R p 4\pi r^2 dr \quad (\text{B.8.14})$$

The total gravitational potential energy Ω of the cloud is given by:

$$\Omega = -\frac{1}{2} \int_0^R \frac{GM(r)\rho(r)}{r} 4\pi r^2 dr \quad (\text{B.8.15})$$

The factor of $\frac{1}{2}$ comes from the fact that the gravitational potential energy is shared between pairs of particles.

Using the equation of hydrostatic equilibrium, we can relate U and Ω :

$$3(\gamma - 1)U + \Omega = 0 \quad (\text{B.8.16})$$

B.8.e (e)

The heat capacity at constant volume C_V for a monoatomic ideal gas is given by $C_V = \frac{3}{2}Nk$, where N is the number of particles and k is the Boltzmann constant. The change in internal energy ΔU when heat Q is added or removed is given by $\Delta U = Q$. Since $U = \frac{3}{2}NkT$, we have $\Delta U = \frac{3}{2}Nk\Delta T$. Setting these equal gives:

$$Q = \frac{3}{2}Nk\Delta T \quad (\text{B.8.17})$$

Solving for ΔT , we get:

$$\Delta T = \frac{2Q}{3Nk} \quad (\text{B.8.18})$$

Since the cloud radiates away energy, Q is negative, so ΔT is also negative, which means the cloud cools down.

The heat capacity of the cloud is given by $C = \frac{Q}{\Delta T}$, so:

$$C = \frac{3}{2}Nk \quad (\text{B.8.19})$$

If two such clouds are exchanging radiant heat, the hotter cloud will cool down and the cooler cloud will heat up until they reach the same temperature. This is due to the second law of thermodynamics, which states that heat flows from hotter to cooler bodies.

B.9 9.

B.9.a (a)

Let: $u = x^{n-1}$ and $dv = xe^{-\alpha x^2} dx$.

$$I_n = \int_0^\infty x^n e^{-\alpha x^2} dx \quad (\text{B.9.1})$$

$$= \left[-\frac{1}{2\alpha} x^{n-1} e^{-\alpha x^2} \right]_0^\infty + \frac{n-1}{2\alpha} \int_0^\infty x^{n-2} e^{-\alpha x^2} dx \quad (\text{B.9.2})$$

$$= 0 + \frac{n-1}{2\alpha} I_{n-2}, \quad (\text{B.9.3})$$

$-\frac{1}{2\alpha} x^{n-1} e^{-\alpha x^2}$ tends to 0 as $x \rightarrow \infty$ and at $x = 0$.

Multiplying both sides by 2α yields the desired result:

$$2\alpha I_n = (n-1)I_{n-2} \quad (\text{B.9.4})$$

can be applied to the velocity distribution function of the monatomic gas,

$$f(\vec{v}) = C \exp(-\alpha[v_x^2 + v_y^2 + v_z^2]). \quad (\text{B.9.5})$$

Since we are dealing with three dimensions (v_x , v_y , and v_z), the integral of the velocity distribution function can be expressed as a triple integral over all velocity space:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(\vec{v}) dv_x dv_y dv_z = 1. \quad (\text{B.9.6})$$

Given the symmetry and independence of the velocity components, this can be rewritten as:

$$\left(\int_{-\infty}^\infty v^n e^{-\alpha v^2} dv \right)^3 = 1. \quad (\text{B.9.7})$$

With a substitution $x = v^2$, we can relate this to our original integral I_n :

$$2^{3/2} \left(\int_0^\infty x^{n/2} e^{-\alpha x} dx \right)^3 = 1, \quad (\text{B.9.8})$$

or equivalently,

$$2^{3/2} I_{n/2}^3 = 1. \quad (\text{B.9.9})$$

B.9.b (b)

We are given the velocity distribution function $f(v) = C \exp(-\alpha[v_x^2 + v_y^2 + v_z^2])$. The constant C is the normalization constant which can be determined by the requirement that the total probability of finding the particle with any velocity is 1:

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) dv_x dv_y dv_z \quad (\text{B.9.10})$$

$$= C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-\alpha[v_x^2 + v_y^2 + v_z^2]) dv_x dv_y dv_z. \quad (\text{B.9.11})$$

This triple integral can be split into the product of three identical single integrals due to the additivity property of the integral:

$$1 = C \left[\int_{-\infty}^{\infty} \exp(-\alpha v_x^2) dv_x \right]^3. \quad (\text{B.9.12})$$

Each integral is of the form $\int_{-\infty}^{\infty} \exp(-\alpha v_i^2) dv_i = \sqrt{\frac{\pi}{\alpha}}$, which can be solved using the Gaussian integral formula:

$$1 = C \left(\sqrt{\frac{\pi}{\alpha}} \right)^3, \quad (\text{B.9.13})$$

Solving for C , we find that

$$C = \left(\frac{\alpha}{\pi} \right)^{3/2}. \quad (\text{B.9.14})$$

This gives the normalization constant C in terms of the parameter α .

$$\alpha = \pi C^{2/3} \quad (\text{B.9.15})$$

B.9.c (c)

The distribution function of the magnitude of the velocity $v = |\vec{v}|$ is the number of states in the velocity space within the shell of thickness dv at the distance v from the origin. In spherical coordinates, this volume is given by $4\pi v^2 dv$. Hence, the speed distribution function, $f(v)$, is obtained by multiplying $f(\vec{v})$ by the volume of this shell, and integrating over the angles:

$$f(v) = 4\pi v^2 f(\vec{v}) \quad (\text{B.9.16})$$

$$= 4\pi v^2 C \exp(-\alpha v^2) \quad (\text{B.9.17})$$

$$= 4\pi v^2 \left(\frac{\alpha}{\pi} \right)^{3/2} \exp(-\alpha v^2). \quad (\text{B.9.18})$$

By substituting $\alpha = \pi C^{2/3}$ from part (b), the distribution function simplifies to:

$$f(v) = 4v^2 C \exp(-\pi C^{2/3} v^2). \quad (\text{B.9.19})$$

function reaches its maximum at a certain speed v_m which we can find by setting the derivative of $f(v)$ with respect to v equal to zero:

$$\frac{df(v)}{dv} = 0 \quad (\text{B.9.20})$$

$$v_m = \frac{1}{\sqrt{\pi C^{2/3}}} \quad (\text{B.9.21})$$

For the sketch we have

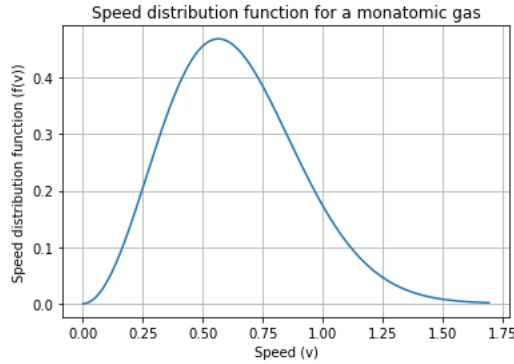


Figure 2: Sketch

B.9.d (d)

$$\langle E \rangle = \int_0^{\infty} E f(v) dv \quad (\text{B.9.22})$$

$$= \int_0^{\infty} \frac{1}{2} m v^2 f(v) dv \quad (\text{B.9.23})$$

$$= \int_0^{\infty} \frac{1}{2} m v^2 4v^2 C \exp(-\pi C^{2/3} v^2) dv. \quad (\text{B.9.24})$$

We can simplify this further by doing a change of variable $x = \pi C^{2/3} v^2$:

$$\langle E \rangle = \frac{2mC}{\pi^{3/2} C} \int_0^{\infty} x^{5/2} \exp(-x) dx. \quad (\text{B.9.25})$$

This integral corresponds to a gamma function, specifically $\Gamma(\frac{7}{2}) = \frac{3}{4}\sqrt{\pi}$.

$$\langle E \rangle = \frac{2mC}{\pi^{3/2}C} \cdot \frac{3}{4}\sqrt{\pi} \quad (\text{B.9.26})$$

$$= \frac{3}{2}mC \quad (\text{B.9.27})$$

$$= \frac{3}{2}m \left(\frac{\alpha}{\pi}\right)^{3/2} \quad (\text{B.9.28})$$

$$= \frac{3}{2}m \left(\frac{k_B T}{m}\right)^{3/2} \quad (\text{B.9.29})$$

$$= \frac{3}{2}k_B T \quad (\text{B.9.30})$$

$$\langle E \rangle = \frac{3}{2}k_B T \quad (\cdot)$$

B.9.e (e)

The Maxwell-Boltzmann distribution also governs the speeds of gas particles during effusion, and hence is applicable to our case. The approach will involve integrating this distribution over the speeds to find the average kinetic energy of the gas particles that effuse through the hole, and hence their final temperature.

Now, let's derive the final temperatures for the monatomic and diatomic gases.

For a gas at equilibrium with temperature T , the Maxwell-Boltzmann distribution function for the speed of a particle is given by:

$$f(v) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-\frac{mv^2}{2k_B T}}, \quad (\text{B.9.31})$$

where m is the mass of the particle, k_B is Boltzmann's constant, T is the temperature, and v is the speed.

We found in part (b) that the speed distribution of the effusing particles is given by:

$$f_e(v) = f(v)v = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^3 e^{-\frac{mv^2}{2k_B T}}. \quad (\text{B.9.32})$$

The average kinetic energy of the effusing particles can be calculated by integrating the product of this function and the kinetic energy $\frac{1}{2}mv^2$ over all speeds:

$$\langle E \rangle = \int_0^\infty \frac{1}{2}mv^2 f_e(v)dv = 2\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^\infty v^5 e^{-\frac{mv^2}{2k_B T}} dv. \quad (\text{B.9.33})$$

The integral can be solved by substituting $u = \frac{mv^2}{2k_B T}$, which leads to $v^2 = \frac{2k_B T}{m}u$ and $v dv = \frac{k_B T}{m} du$. The limits of integration remain the same because when $v = 0$ we have $u = 0$, and when $v = \infty$ we have $u = \infty$. Therefore:

$$\langle E \rangle = 2\pi \left(\frac{m}{2\pi k_B T} \right)^{3/2} \frac{2k_B^3 T^3}{m^3} \int_0^\infty u^2 e^{-u} du = \frac{3}{2} k_B T, \quad (\text{B.9.34})$$

where the integral of $u^2 e^{-u}$ from 0 to ∞ is equal to 2

The final temperature T_f of the effusing particles can be found from the equation $\langle E \rangle = \frac{3}{2} k_B T_f$, which gives $T_f = T$.

Therefore, for both the monatomic and diatomic gases, the final temperature of the gas particles that have effused into the box is the same as the initial temperature of the gas in the container, assuming that the effusion process is adiabatic and the hole's radius is larger than the mean free path of the gas particles.

B.10 10

B.10.a (a)

We are given a system of independent paramagnetic atoms each with a dipole moment μ that can be in one of two states: aligned with the magnetic field (μB) or against the magnetic field ($-\mu B$). We can calculate the relative probabilities of these states using the Boltzmann factor.

The partition function for each atom is then:

$$Z = e^{\beta\mu B} + e^{-\beta\mu B}, \quad (\text{B.10.1})$$

where $\beta = \frac{1}{k_B T}$, k_B is the Boltzmann constant, T is the temperature and B is the magnetic field strength.

The magnetization per atom m (average dipole moment) can be obtained by taking the derivative of the free energy $F = -k_B T \ln(Z)$ with respect to the magnetic field strength:

$$m = \frac{\partial F}{\partial B} = -\mu \tanh(\beta\mu B), \quad (\text{B.10.2})$$

The magnetization per unit volume M is simply given by $n \cdot m$, where n is the number of atoms per unit volume. Hence,

$$M = -n\mu \tanh(\beta\mu B). \quad (\text{B.10.3})$$

We can then write the susceptibility χ , which is defined as $\chi = \frac{\partial M}{\partial B}$, and after differentiating the above expression with respect to B , we get:

$$\chi = n\mu^2 \beta \operatorname{sech}^2(\beta\mu B). \quad (\text{B.10.4})$$

In the limit of small B (or high temperatures), $\operatorname{sech}^2(x) \approx 1$, hence the susceptibility simplifies to:

$$\chi = n\mu^2 \beta. \quad (\text{B.10.5})$$

B.10.b (b)

Given that each atom can be in one of two states (aligned or anti-aligned with the field), we can find the total number of ways to arrange all the atoms (the number of microstates W) using the binomial coefficient:

$$W = \frac{N!}{N_+!N_-!} \quad (\text{B.10.6})$$

We then relate this to entropy via Boltzmann's equation:

$$S = k_B \ln W \quad (\text{B.10.7})$$

To express S in terms of m , we first express N_+ and N_- in terms of the probabilities p_+ and p_- , respectively, which we have from part (a). Specifically, $N_+ = Np_+$ and $N_- = Np_-$, where N is the total number of atoms.

Next, we note that the total dipole moment per unit volume m is related to N_+ and N_- by:

$$N_+ - N_- = N \tanh(\beta\mu B) = \frac{m}{\mu} \quad (\text{B.10.8})$$

Substituting this into our equation for W , we get:

$$W = \frac{N!}{\left(\frac{N}{2} + \frac{m}{2\mu}\right)! \left(\frac{N}{2} - \frac{m}{2\mu}\right)!} \quad (\text{B.10.9})$$

Finally, we use Stirling's approximation to simplify the factorial terms, giving us an expression for S in terms of m :

$$S \approx k_B \left[N \ln N - \left(\frac{N}{2} + \frac{m}{2\mu}\right) \ln \left(\frac{N}{2} + \frac{m}{2\mu}\right) - \left(\frac{N}{2} - \frac{m}{2\mu}\right) \ln \left(\frac{N}{2} - \frac{m}{2\mu}\right) \right] \quad (\text{B.10.10})$$

This is the relationship between entropy S and the total dipole moment m of the solid.

B.10.c (c)

At very low temperatures ($T \rightarrow 0$), the magnetic dipoles align with the magnetic field to minimize their energy. This results in a low-entropy, ordered state. Any small increase in temperature introduces significant disorder into the system, and consequently, the heat capacity is high at very low temperatures.

At very high temperatures ($T \rightarrow \infty$), the magnetic dipoles are randomly oriented as the thermal energy surpasses the energy of the magnetic field. This creates a high-entropy, disordered state. Therefore, the heat capacity is low at high temperatures since changes in temperature don't significantly affect the system's order.

There's a peak in the heat capacity at some intermediate temperature where the system transitions from the low-temperature ordered state to the high-temperature disordered state.

The integral $\int_0^\infty \frac{C}{T} dT$ is related to the total entropy change in the system when it is heated from absolute zero to infinite temperature. According to the third law of thermodynamics and Debye's law, in a paramagnetic system, this total change is $Nk_B \ln 2$ (where N is the number of dipoles and k_B is the Boltzmann constant). Hence, $\int_0^\infty \frac{C}{T} dT = Nk_B \ln 2$

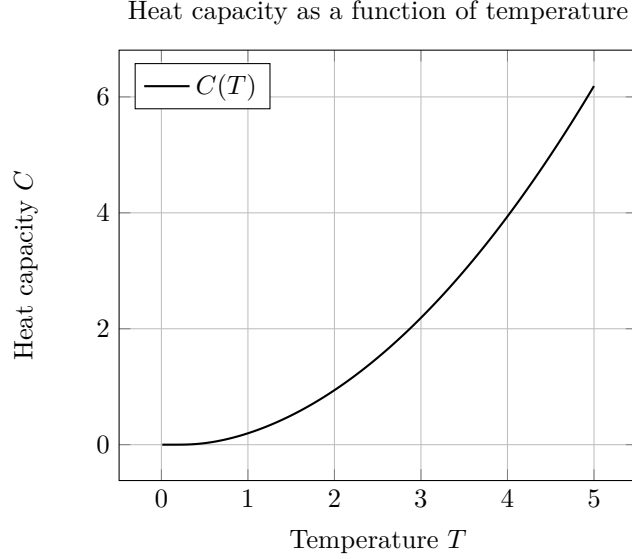


Figure 3: Heat capacity C as a function of temperature

B.10.d (d)

In an adiabatic process, entropy remains constant. Therefore, we can write $S(B_1, T_1) = S(B_2, T_2)$ where S is the entropy, B is the magnetic field, and T is the temperature. From our earlier computations, we have a relationship between entropy S and magnetization M , which is proportional to the dipole moment per unit volume m . The dipoles in the system would try to align with the new field direction when the magnetic field is reduced adiabatically. This is due to the fact that the aligned state is energetically favourable compared to the anti-aligned state. Now, the entropy of the system remains constant in an adiabatic process. By applying this condition and rearranging, we can compute the final temperature T_2 .

Given $S(B_1, T_1) = S(B_2, T_2)$, we have

$$Nk_B \left[\frac{1}{2} \ln (2 \cosh (\beta_1 \mu B_1)) \right] = Nk_B \left[\frac{1}{2} \ln (2 \cosh (\beta_2 \mu B_2)) \right] \quad (\text{B.10.11})$$

Here, $\beta_1 = \frac{1}{k_B T_1}$, $\beta_2 = \frac{1}{k_B T_2}$, $B_1 = 1$ T, and $B_2 = 0.3$ T.

We can cancel out the common factors and simplify to get:

$$\ln \left(2 \cosh \left(\frac{\mu B_1}{T_1} \right) \right) = \ln \left(2 \cosh \left(\frac{\mu B_2}{T_2} \right) \right) \quad (\text{B.10.12})$$

Therefore,

$$\frac{\mu B_1}{T_1} = \frac{\mu B_2}{T_2} \quad (\text{B.10.13})$$

We can solve this equation to get T_2 , the final temperature:

$$T_2 = \frac{B_2}{B_1} T_1 = \frac{0.3}{1} \cdot 1K = 0.3K \quad (\text{B.10.14})$$

Thus, when the magnetic field is reduced adiabatically from $1T$ to $0.3T$, the final temperature of the solid will be $0.3K$.

B.10.e (e)

Let's consider the case of a container of gas at temperature T which is streaming particles into a vacuum through a hole whose radius is small compared to the mean free path of the particles. After a short time, a box is opened which catches some of these particles, and we want to find the final temperature of the gas trapped in this box.

Given that the hole is small compared to the mean free path, we can say that only the particles with a component of velocity directed towards the hole can escape the container. In the case of a Maxwell-Boltzmann distribution of velocities, this corresponds to the most probable speed, $v_p = \sqrt{\frac{2k_B T}{m}}$, where k_B is Boltzmann's constant, T is the temperature of the gas, and m is the mass of the gas particles.

(i) For a monatomic gas, each atom has 3 degrees of freedom, so the total energy per particle is $\frac{3}{2}k_B T$. After the gas escapes into the vacuum, the average energy per particle does not change, as it is an adiabatic process. The final temperature T_f of the gas trapped in the box can be found from the relation $\frac{3}{2}k_B T = \frac{3}{2}k_B T_f$, so $T_f = T$.

(ii) For a diatomic gas in which rotation is fully thermally excited and vibration is not excited, each molecule has 5 degrees of freedom, so the total energy per molecule is $\frac{5}{2}k_B T$. Similarly, the average energy per molecule does not change when the gas escapes into the vacuum. The final temperature T_f of the gas trapped in the box can be found from the relation $\frac{5}{2}k_B T = \frac{5}{2}k_B T_f$, so $T_f = T$.

In a similar experiment with a hole of radius larger than the mean free path, we would expect the final temperature of the gas trapped in the box to be lower. This is because all particles, regardless of their speed, would have a chance to escape, so the average energy of the escaping particles would be lower.